Approximate Capacity of the Gaussian Interference Channel with Noisy Channel-Output Feedback

Victor Quintero, Samir M. Perlaza, Iñaki Esnaola and Jean-Marie Gorce

***Abstract*—In this paper, an achievability region and a converse region for the two-user Gaussian interference channel with noisy channel-output feedback (G-IC-NOF) are presented. The achievability region is obtained using a random coding argument and three well-known techniques: rate splitting, superposition coding and backward decoding. The converse region is obtained using some of the existing perfect-output feedback outer-bounds as well as a set of new outer-bounds that are obtained by using genie-aided models of the original G-IC-NOF. Finally, it is shown that the achievability region and the converse region approximate the capacity region of the G-IC-NOF to within a constant gap in bits per channel use.**

***Index Terms*—Capacity, Interference Channel, Noisy Channel-Output Feedback.**

I. NOTATION

Throughout this paper, $(\cdot)^+$ denotes the positive part operator, i.e., $(\cdot)^+ = \max(\cdot, 0)$ and $\mathbb{E}_X[\cdot]$ denotes the expectation with respect to the distribution of the random variable X. The logarithm function $\log$ is assumed to be base 2.

II. SYSTEM MODEL

Consider the two-user G-IC-NOF in Figure 1. Transmitter i, with $i \in \{1,2\}$, communicates with receiver i subject to the interference produced by transmitter j, with $j \in \{1,2\}\backslash\{i\}$. There are two independent and uniformly distributed messages, $W_i \in \mathcal{W}_i$, with $\mathcal{W}_i = \{1, 2, \dots, 2^{NR_i}\}$, where N denotes the block-length in channel uses and R_i is the transmission rate in bits per channel use. At each block, transmitter i sends the codeword $\boldsymbol{X}_i = (X_{i,1}, X_{i,2}, \dots, X_{i,N})^{\mathsf{T}} \in \mathcal{X}_i^N$, where $\mathcal{X}_i$ and $\mathcal{X}_i^N$ are respectively the channel-input alphabet and the codebook of transmitter i.

The channel coefficient from transmitter j to receiver i is denoted by h_{ij}; the channel coefficient from transmitter i to receiver i is denoted by $\overrightarrow{h}_{ii}$; and the channel coefficient from channel-output i to transmitter i is denoted by $\overleftarrow{h}_{ii}$. All channel coefficients are assumed to be non-negative real numbers. At a given channel use $n \in \{1, 2, \dots, N\}$, the channel output at receiver i is denoted by $\overrightarrow{Y}_{i,n}$. During channel use n, the input-output relation of the channel model is given by

$$\overrightarrow{Y}_{i,n} = \overrightarrow{h}_{ii} X_{i,n} + h_{ij} X_{j,n} + \overrightarrow{Z}_{i,n}, \tag{1}$$

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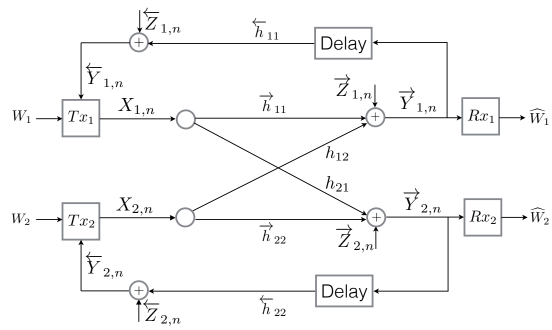

Fig. 1. Gaussian interference channel with noisy channel-output feedback at channel use n.

where $\overrightarrow{Z}_{i,n}$ is a real Gaussian random variable with zero mean and unit variance that represents the noise at the input of receiver i. Let $d > 0$ be the finite feedback delay measured in channel uses. At the end of channel use n, transmitter i observes $\overleftarrow{Y}_{i,n}$, which consists of a scaled and noisy version of $\overrightarrow{Y}_{i,n-d}$. More specifically,

$$\overleftarrow{Y}_{i,n} = \begin{cases} \overleftarrow{Z}_{i,n} & \text{for } n \in \{1,2,\dots,d\} \\ \overleftarrow{h}_{ii} \overrightarrow{Y}_{i,n-d} + \overleftarrow{Z}_{i,n}, & \text{for } n \in \{d+1, d+2, \dots, N\}, \end{cases} \tag{2}$$

where $\overleftarrow{Z}_{i,n}$ is a real Gaussian random variable with zero mean and unit variance that represents the noise in the feedback link of transmitter-receiver pair i. The random variables $\overrightarrow{Z}_{i,n}$ and $\overleftarrow{Z}_{i,n}$ are independent and identically distributed. In the following, without loss of generality, the feedback delay is assumed to be one channel use, i.e., $d = 1$. The encoder of transmitter i is defined by a set of deterministic functions $f_i^{(1)}, \dots, f_i^{(N)}$, with $f_i^{(1)} : \mathcal{W}_i \rightarrow \mathcal{X}_i$ and for all $n \in \{2, \dots, N\}$, $f_i^{(n)} : \mathcal{W}_i \times \mathbb{R}^{n-1} \rightarrow \mathcal{X}_i$, such that

$$X_{i,1} = f_i^{(1)}(W_i), \text{ and} \tag{3a}$$

$$X_{i,n} = f_i^{(n)}\left(W_i, \overleftarrow{Y}_{i,1}, \dots, \overleftarrow{Y}_{i,n-1}\right). \tag{3b}$$

The components of the input vector $\boldsymbol{X}_i$ are real numbers subject to an average power constraint:

$$\frac{1}{N} \sum_{n=1}^{N} \mathbb{E}\left(X_{i,n}^2\right) \leq 1, \tag{4}$$

where the expectation is taken over the joint distribution of the message indexes W_1, W_2, and the noise terms, i.e., $\overrightarrow{Z}_1$, $\overrightarrow{Z}_2$, $\overleftarrow{Z}_1$, and $\overleftarrow{Z}_2$. The dependence of $X_{i,n}$ on W_1, W_2, and the previously observed noise realizations is due to the effect of feedback as shown in (2) and (3).

Assume that during a given communication, T blocks are transmitted. Hence, the decoder of receiver i is defined by a deterministic function $\psi_i : \mathbb{R}_i^{NT} \rightarrow \mathcal{W}_i^{T}$. At the end of the communication, receiver i uses the vector $\left(\overrightarrow{Y}_{i,1}, \overrightarrow{Y}_{i,2}, \ldots, \overrightarrow{Y}_{i,NT}\right)^{\mathsf{T}}$ to obtain an estimate of the message indices

$$\left(\widehat{W}_i^{(1)}, \widehat{W}_i^{(2)}, \ldots, \widehat{W}_i^{(T)}\right) = \psi_i\left(\overrightarrow{Y}_{i,1}, \overrightarrow{Y}_{i,2}, \ldots, \overrightarrow{Y}_{i,NT}\right), \quad (5)$$

where $\widehat{W}_i^{(t)}$ is an estimate of the message index sent during block $t \in \{1, 2, \ldots, T\}$. The decoding error probability in the two-user G-IC-NOF during block t of a codebook of block-length N, denoted by $P_e^{(t)}(N)$, is given by

$$P_e^{(t)}(N) = \max\left(\Pr\left[\widehat{W_1}^{(t)} \neq W_1^{(t)}\right], \Pr\left[\widehat{W_2}^{(t)} \neq W_2^{(t)}\right]\right). \quad (6)$$

The definition of an achievable rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ is given below.

Definition 1 (Achievable Rate Pairs): A rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ *is achievable if there exists at least one pair of codebooks* $\mathcal{X}_1^N$ *and* $\mathcal{X}_2^N$ *with codewords of length* N*, and the corresponding encoding functions* $f_1^{(1)}, \ldots, f_1^{(N)}$ *and* $f_2^{(1)}, \ldots, f_2^{(N)}$ *such that the decoding error probability* $P_e^{(t)}(N)$ *can be made arbitrarily small by letting the block-length* N *grow to infinity, for all blocks* $t \in \{1, \ldots, T\}$.

The two-user G-IC-NOF in Figure 1 can be fully described by six parameters: $\overrightarrow{\mathrm{SNR}}_i$, $\overleftarrow{\mathrm{SNR}}_i$, and INR_{ij}, with $i \in \{1, 2\}$ and $j \in \{1, 2\} \backslash \{i\}$, which are defined as follows:

$$\overrightarrow{\mathrm{SNR}}_i = \overrightarrow{h}_{ii}^{2}, \quad (7)$$

$$\mathrm{INR}_{ij} = h_{ij}^2 \text{ and} \quad (8)$$

$$\overleftarrow{\mathrm{SNR}}_i = \overleftarrow{h}_{ii}^{2}\left(\overrightarrow{h}_{ii}^{2} + 2\overrightarrow{h}_{ii} h_{ij} + h_{ij}^2 + 1\right). \quad (9)$$

III. Main Results

This section introduces an achievable region (Theorem 1) and a converse region (Theorem 2), denoted by $\underline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$ and $\overline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$ respectively, for the two-user G-IC-NOF with fixed parameters $\overrightarrow{\mathrm{SNR}}_1$, $\overrightarrow{\mathrm{SNR}}_2$, INR_{12}, INR_{21}, $\overleftarrow{\mathrm{SNR}}_1$, and $\overleftarrow{\mathrm{SNR}}_2$. In general, the capacity region of a given multi-user channel is said to be approximated to within a constant gap according to the following definition.

Definition 2 (Approximation to within ξ units): A closed and convex set $\mathcal{T} \subset \mathbb{R}_+^m$ *is approximated to within* ξ *units by the sets* $\underline{\mathcal{T}}$ *and* $\overline{\mathcal{T}}$ *if* $\underline{\mathcal{T}} \subseteq \mathcal{T} \subseteq \overline{\mathcal{T}}$ *and for all* $\boldsymbol{t} = (t_1, \ldots, t_m) \in \overline{\mathcal{T}}$ *then* $((t_1 - \xi)^+, \ldots, (t_m - \xi)^+) \in \underline{\mathcal{T}}$.

Denote by $\mathcal{C}_{\mathrm{GIC-NOF}}$ the capacity region of the 2-user G-IC-NOF. The achievable region $\underline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$ and the converse region $\overline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$ approximate the capacity region $\mathcal{C}_{\mathrm{GIC-NOF}}$ to within 4.4 bits per channel use (Theorem 3).

A. An Achievable Region for the Two-User G-IC-NOF

The description of the achievable region $\underline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$ is presented using the constants $a_{1,i}$; the functions $a_{2,i} : [0,1] \rightarrow \mathbb{R}_+$, $a_{l,i} : [0,1]^2 \rightarrow \mathbb{R}_+$, with $l \in \{3, \ldots, 6\}$; and $a_{7,i} : [0,1]^3 \rightarrow \mathbb{R}_+$, which are defined as follows, for all $i \in \{1, 2\}$, with $j \in \{1, 2\} \setminus \{i\}$:

$$a_{1,i} = \frac{1}{2}\log\left(2 + \frac{\overrightarrow{\mathrm{SNR}}_i}{\mathrm{INR}_{ji}}\right) - \frac{1}{2}, \quad (10a)$$

$$a_{2,i}(\rho) = \frac{1}{2}\log\Big(b_{1,i}(\rho) + 1\Big) - \frac{1}{2}, \quad (10b)$$

$$a_{3,i}(\rho,\mu) = \frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_i\Big(b_{2,i}(\rho) + 2\Big) + b_{1,i}(1) + 1}{\overleftarrow{\mathrm{SNR}}_i\Big((1-\mu)b_{2,i}(\rho) + 2\Big) + b_{1,i}(1) + 1}\right), \quad (10c)$$

$$a_{4,i}(\rho,\mu) = \frac{1}{2}\log\Big(\Big(1-\mu\Big)b_{2,i}(\rho) + 2\Big) - \frac{1}{2}, \quad (10d)$$

$$a_{5,i}(\rho,\mu) = \frac{1}{2}\log\left(2 + \frac{\overrightarrow{\mathrm{SNR}}_i}{\mathrm{INR}_{ji}} + \Big(1-\mu\Big)b_{2,i}(\rho)\right) - \frac{1}{2}, \quad (10e)$$

$$a_{6,i}(\rho,\mu) = \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_i}{\mathrm{INR}_{ji}}\Big(\Big(1-\mu\Big)b_{2,j}(\rho) + 1\Big) + 2\right) - \frac{1}{2}, \quad (10f)$$

and

$$a_{7,i}(\rho,\mu_1,\mu_2) = \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_i}{\mathrm{INR}_{ji}}\Big(\Big(1-\mu_i\Big)b_{2,j}(\rho) + 1\Big) + \Big(1-\mu_j\Big)b_{2,i}(\rho) + 2\right) - \frac{1}{2}, \quad (10g)$$

where the functions $b_{l,i} : [0,1] \rightarrow \mathbb{R}_+$, with $(l,i) \in \{1,2\}^2$ are defined as follows:

$$b_{1,i}(\rho) = \overrightarrow{\mathrm{SNR}}_i + 2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_i \mathrm{INR}_{ij}} + \mathrm{INR}_{ij} \text{ and} \quad (11a)$$

$$b_{2,i}(\rho) = \Big(1-\rho\Big)\mathrm{INR}_{ij} - 1, \quad (11b)$$

with $j \in \{1, 2\} \setminus \{i\}$.

Note that the functions in (10) and (11) depend on $\overrightarrow{\mathrm{SNR}}_1$, $\overrightarrow{\mathrm{SNR}}_2$, INR_{12}, INR_{21}, $\overleftarrow{\mathrm{SNR}}_1$, and $\overleftarrow{\mathrm{SNR}}_2$, however as these parameters are fixed in this analysis, this dependence is not emphasized in the definition of these functions. Finally, using this notation, Theorem 1 is presented on the next page.

Proof: The proof of Theorem 1 is presented in [1]. ■

B. Comments on the Achievability

The achievable region is obtained using a random coding argument and combining three classical tools: rate splitting, superposition coding, and backward decoding. This coding scheme is described in [1] and it is specially designed for the two-user IC-NOF. Consequently, only the strictly needed number of superposition code-layers is used. Other achievable schemes, as reported in [2], can also be obtained as special cases of the more general scheme presented in [3]. However, in this more general case, the resulting code for the IC-NOF contains a handful of unnecessary superposing code-layers, which complicates the error probability analysis.

C. A Converse Region for the Two-User G-IC-NOF

The description of the converse region $\overline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$ is determined by the ratios $\frac{\mathrm{INR}_{ij}}{\overrightarrow{\mathrm{SNR}}_j}$, and $\frac{\mathrm{INR}_{ji}}{\overrightarrow{\mathrm{SNR}}_j}$, for all $i \in \{1, 2\}$, with $j \in \{1, 2\} \setminus \{i\}$. All relevant scenarios regarding these ratios

Theorem 1: The capacity region $\mathcal{C}_{\mathrm{GIC-NOF}}$ contains the region $\underline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$ given by the closure of the set of all possible non-negative achievable rate pairs (R_1, R_2) that satisfy

$$R_1 \leqslant \min\Big(a_{2,1}(\rho), a_{6,1}(\rho,\mu_1)+a_{3,2}(\rho,\mu_1), a_{1,1}+a_{3,2}(\rho,\mu_1)+a_{4,2}(\rho,\mu_1)\Big), \quad (12a)$$

$$R_2 \leqslant \min\Big(a_{2,2}(\rho), a_{3,1}(\rho,\mu_2)+a_{6,2}(\rho,\mu_2), a_{3,1}(\rho,\mu_2)+a_{4,1}(\rho,\mu_2)+a_{1,2}\Big), \quad (12b)$$

$$R_1+R_2 \leqslant \min\Big(a_{2,1}(\rho)+a_{1,2}, a_{1,1}+a_{2,2}(\rho), a_{3,1}(\rho,\mu_2)+a_{1,1}+a_{3,2}(\rho,\mu_1)+a_{7,2}(\rho,\mu_1,\mu_2),$$
$$a_{3,1}(\rho,\mu_2)+a_{5,1}(\rho,\mu_2)+a_{3,2}(\rho,\mu_1)+a_{5,2}(\rho,\mu_1), a_{3,1}(\rho,\mu_2)+a_{7,1}(\rho,\mu_1,\mu_2)+a_{3,2}(\rho,\mu_1)+a_{1,2}\Big), \quad (12c)$$

$$2R_1+R_2 \leqslant \min\Big(a_{2,1}(\rho)+a_{1,1}+a_{3,2}(\rho,\mu_1)+a_{7,2}(\rho,\mu_1,\mu_2), \quad (12d)$$
$$a_{3,1}(\rho,\mu_2)+a_{1,1}+a_{7,1}(\rho,\mu_1,\mu_2)+2a_{3,2}(\rho,\mu_1)+a_{5,2}(\rho,\mu_1), a_{2,1}(\rho)+a_{1,1}+a_{3,2}(\rho,\mu_1)+a_{5,2}(\rho,\mu_1)\Big),$$

$$R_1+2R_2 \leqslant \min\Big(a_{3,1}(\rho,\mu_2)+a_{5,1}(\rho,\mu_2)+a_{2,2}(\rho)+a_{1,2}, a_{3,1}(\rho,\mu_2)+a_{7,1}(\rho,\mu_1,\mu_2)+a_{2,2}(\rho)+a_{1,2},$$
$$2a_{3,1}(\rho,\mu_2)+a_{5,1}(\rho,\mu_2)+a_{3,2}(\rho,\mu_1)+a_{1,2}+a_{7,2}(\rho,\mu_1,\mu_2)\Big), \quad (12e)$$

with $(\rho,\mu_1,\mu_2) \in \left[0, \left(1-\max\left(\frac{1}{\mathrm{INR}_{12}}, \frac{1}{\mathrm{INR}_{21}}\right)\right)^+\right] \times [0,1] \times [0,1]$.

are described by two events denoted by $S_{l_1,1}$ and $S_{l_2,2}$, where $(l_1, l_2) \in \{1,\ldots,5\}^2$. The events are defined as follows:

$$S_{1,i}:\ \overrightarrow{\mathrm{SNR}}_j < \min\left(\mathrm{INR}_{ij}, \mathrm{INR}_{ji}\right), \quad (13a)$$
$$S_{2,i}:\ \mathrm{INR}_{ji} \leqslant \overrightarrow{\mathrm{SNR}}_j < \mathrm{INR}_{ij}, \quad (13b)$$
$$S_{3,i}:\ \mathrm{INR}_{ij} \leqslant \overrightarrow{\mathrm{SNR}}_j < \mathrm{INR}_{ji}, \quad (13c)$$
$$S_{4,i}:\ \max\left(\mathrm{INR}_{ij}, \mathrm{INR}_{ji}\right) \leqslant \overrightarrow{\mathrm{SNR}}_j < \mathrm{INR}_{ij}\mathrm{INR}_{ji}, \quad (13d)$$
$$S_{5,i}:\ \overrightarrow{\mathrm{SNR}}_j \geqslant \max\left(\mathrm{INR}_{ij}, \mathrm{INR}_{ji}, \mathrm{INR}_{ij}\mathrm{INR}_{ji}\right). \quad (13e)$$

Note that for all $i \in \{1,2\}$, the events $S_{1,i}$, $S_{2,i}$, $S_{3,i}$, $S_{4,i}$, and $S_{5,i}$ are mutually exclusive. This observation shows that given any 4-tuple $(\overrightarrow{\mathrm{SNR}}_1, \overrightarrow{\mathrm{SNR}}_2, \mathrm{INR}_{12}, \mathrm{INR}_{21})$, there always exists one and only one pair of events $(S_{l_1,1}, S_{l_2,2})$, with $(l_1,l_2) \in \{1,\ldots,5\}^2$, that identifies a unique scenario. Note also that the pairs of events $(S_{2,1}, S_{2,2})$ and $(S_{3,1}, S_{3,2})$ are not feasible. In view of this, twenty-three different scenarios can be identified using the events in (13). Once the exact scenario is identified, the converse region is described using the functions $\kappa_{l,i} : [0,1] \to \mathbb{R}_+$, with $(l,i) \in \{1,\ldots,3\} \times \{1,2\}$; $\kappa_l : [0,1] \to \mathbb{R}_+$, with $l \in \{4,5\}$; $\kappa_{6,l} : [0,1] \to \mathbb{R}_+$, with $l \in \{1,\ldots,4\}$; and $\kappa_{7,i,l} : [0,1] \to \mathbb{R}_+$, with $(i,l) \in \{1,2\}^2$. These functions are defined as follows for all $i \in \{1,2\}$, with $j \in \{1,2\} \setminus \{i\}$:

$$\kappa_{1,i}(\rho) = \frac{1}{2}\log\Big(b_{1,i}(\rho)+1\Big), \quad (14a)$$

$$\kappa_{2,i}(\rho) = \frac{1}{2}\log\Big(1+b_{5,j}(\rho)\Big) + \frac{1}{2}\log\left(1+\frac{b_{4,i}(\rho)}{1+b_{5,j}(\rho)}\right), \quad (14b)$$

$$\kappa_{3,i}(\rho) = \frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_j\Big(b_{4,i}(\rho)+b_{5,j}(\rho)+1\Big)}{\Big(b_{1,j}(1)+1\Big)\Big(b_{4,i}(\rho)+1\Big)}+1\right) + \frac{1}{2}\log\Big(b_{4,i}(\rho)+1\Big), \quad (14c)$$

$$\kappa_4(\rho) = \frac{1}{2}\log\left(1+\frac{b_{4,1}(\rho)}{1+b_{5,2}(\rho)}\right) + \frac{1}{2}\log\Big(b_{1,2}(\rho)+1\Big), \quad (14d)$$

$$\kappa_5(\rho) = \frac{1}{2}\log\left(1+\frac{b_{4,2}(\rho)}{1+b_{5,1}(\rho)}\right) + \frac{1}{2}\log\Big(b_{1,1}(\rho)+1\Big), \quad (14e)$$

$$\kappa_6(\rho) = \begin{cases} \kappa_{6,1}(\rho) & \text{if } (S_{1,2} \vee S_{2,2} \vee S_{5,2}) \\ & \quad \wedge (S_{1,1} \vee S_{2,1} \vee S_{5,1}) \\ \kappa_{6,2}(\rho) & \text{if } (S_{1,2} \vee S_{2,2} \vee S_{5,2}) \\ & \quad \wedge (S_{3,1} \vee S_{4,1}) \\ \kappa_{6,3}(\rho) & \text{if } (S_{3,2} \vee S_{4,2}) \\ & \quad \wedge (S_{1,1} \vee S_{2,1} \vee S_{5,1}) \\ \kappa_{6,4}(\rho) & \text{if } (S_{3,2} \vee S_{4,2}) \wedge (S_{3,1} \vee S_{4,1}) \end{cases} \quad (14f)$$

$$\kappa_{7,i}(\rho) = \begin{cases} \kappa_{7,i,1}(\rho) & \text{if } (S_{1,i} \vee S_{2,i} \vee S_{5,i}) \\ \kappa_{7,i,2}(\rho) & \text{if } (S_{3,i} \vee S_{4,i}) \end{cases} \quad (14g)$$

where

$$\kappa_{6,1}(\rho) = \frac{1}{2}\log\Big(b_{1,1}(\rho)+b_{5,1}(\rho)\mathrm{INR}_{21}\Big) - \frac{1}{2}\log\Big(1+\mathrm{INR}_{12}\Big) + \frac{1}{2}\log\left(1+\frac{b_{5,2}(\rho)\overleftarrow{\mathrm{SNR}}_2}{b_{1,2}(1)+1}\right) + \frac{1}{2}\log\Big(b_{1,2}(\rho)+b_{5,1}(\rho)\mathrm{INR}_{21}\Big) - \frac{1}{2}\log\Big(1+\mathrm{INR}_{21}\Big) + \frac{1}{2}\log\left(1+\frac{b_{5,1}(\rho)\overleftarrow{\mathrm{SNR}}_1}{b_{1,1}(1)+1}\right) + \log(2\pi e), \quad (15a)$$

$$\kappa_{6,2}(\rho) = \frac{1}{2}\log\left(b_{6,2}(\rho)+\frac{b_{5,1}(\rho)\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}\Big(\overrightarrow{\mathrm{SNR}}_2+b_{3,2}\Big)\right) - \frac{1}{2}\log\Big(1+\mathrm{INR}_{12}\Big) + \frac{1}{2}\log\left(1+\frac{b_{5,1}(\rho)\overleftarrow{\mathrm{SNR}}_1}{b_{1,1}(1)+1}\right) + \frac{1}{2}\log\Big(b_{1,1}(\rho)+b_{5,1}(\rho)\mathrm{INR}_{21}\Big) - \frac{1}{2}\log\Big(1+\mathrm{INR}_{21}\Big) + \frac{1}{2}\log\left(1+\frac{b_{5,2}(\rho)}{\overrightarrow{\mathrm{SNR}}_2}\left(\mathrm{INR}_{12}+\frac{b_{3,2}\overleftarrow{\mathrm{SNR}}_2}{b_{1,2}(1)+1}\right)\right) - \frac{1}{2}\log\left(1+\frac{b_{5,1}(\rho)\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}\right) + \log(2\pi e), \quad (15b)$$

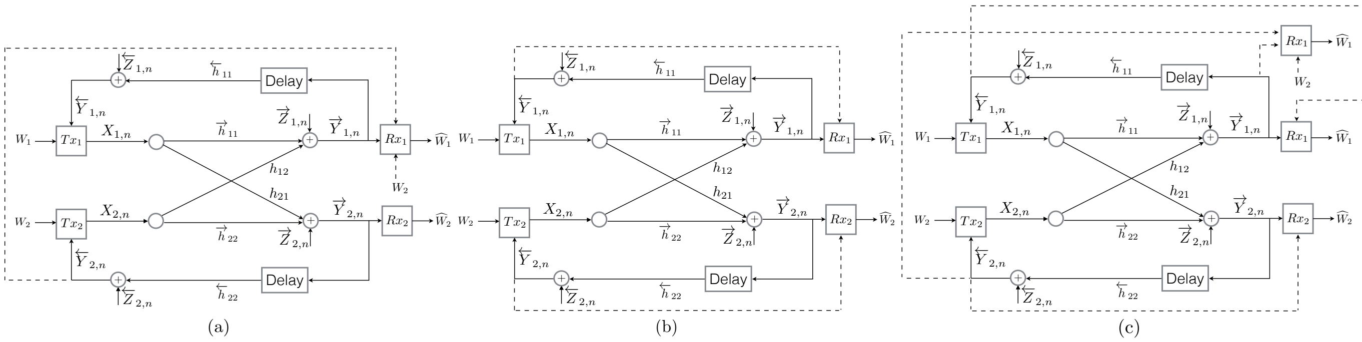

Fig. 2. Genie-Aided G-IC-NOF models for channel use n. (a) Model used to calculate the outer-bound on R_1; (b) Model used to calculate the outer-bound on $R_1 + R_2$; and (c) Model used to calculate the outer-bound on $2R_1 + R_2$

$$\begin{aligned}\kappa_{6,3}(\rho)=&\frac{1}{2}\log\left(b_{6,1}(\rho)+\frac{b_{5,1}(\rho)\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}\left(\overrightarrow{\mathrm{SNR}}_1+b_{3,1}\right)\right)\\&-\frac{1}{2}\log\left(1+\mathrm{INR}_{12}\right)+\frac{1}{2}\log\left(1+\frac{b_{5,2}(\rho)\overleftarrow{\mathrm{SNR}}_2}{b_{1,2}(1)+1}\right)\\&+\frac{1}{2}\log\left(b_{1,2}(\rho)+b_{5,1}(\rho)\mathrm{INR}_{21}\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{21}\right)\\&+\frac{1}{2}\log\left(1+\frac{b_{5,1}(\rho)}{\overrightarrow{\mathrm{SNR}}_1}\left(\mathrm{INR}_{21}+\frac{b_{3,1}\overleftarrow{\mathrm{SNR}}_1}{b_{1,1}(1)+1}\right)\right)\\&-\frac{1}{2}\log\left(1+\frac{b_{5,1}(\rho)\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}\right)+\log(2\pi e),\end{aligned}\tag{15c}$$

$$\begin{aligned}\kappa_{6,4}(\rho)=&\frac{1}{2}\log\left(b_{6,1}(\rho)+\frac{b_{5,1}(\rho)\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}\left(\overrightarrow{\mathrm{SNR}}_1+b_{3,1}\right)\right)\\&-\frac{1}{2}\log\left(1+\mathrm{INR}_{12}\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{21}\right)\\&+\frac{1}{2}\log\left(1+\frac{b_{5,2}(\rho)}{\overrightarrow{\mathrm{SNR}}_2}\left(\mathrm{INR}_{12}+\frac{b_{3,2}\overleftarrow{\mathrm{SNR}}_2}{b_{1,2}(1)+1}\right)\right)\\&-\frac{1}{2}\log\left(1+\frac{b_{5,1}(\rho)\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}\right)\\&-\frac{1}{2}\log\left(1+\frac{b_{5,1}(\rho)\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}\right)\\&+\frac{1}{2}\log\left(b_{6,2}(\rho)+\frac{b_{5,1}(\rho)\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}\left(\overrightarrow{\mathrm{SNR}}_2+b_{3,2}\right)\right)\\&+\frac{1}{2}\log\left(1+\frac{b_{5,1}(\rho)}{\overrightarrow{\mathrm{SNR}}_1}\left(\mathrm{INR}_{21}+\frac{b_{3,1}\overleftarrow{\mathrm{SNR}}_1}{b_{1,1}(1)+1}\right)\right)\\&+\log(2\pi e),\end{aligned}\tag{15d}$$

and

$$\begin{aligned}\kappa_{7,i,1}(\rho)=&\frac{1}{2}\log\left(b_{1,i}(\rho)+1\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{ij}\right)\\&+\frac{1}{2}\log\left(1+\frac{b_{5,j}(\rho)\overleftarrow{\mathrm{SNR}}_j}{b_{1,j}(1)+1}\right)\\&+\frac{1}{2}\log\left(b_{1,j}(\rho)+b_{5,i}(\rho)\mathrm{INR}_{ji}\right)\\&+\frac{1}{2}\log\left(1+b_{4,i}(\rho)+b_{5,j}(\rho)\right)-\frac{1}{2}\log\left(1+b_{5,j}(\rho)\right)\\&+2\log(2\pi e),\end{aligned}\tag{16a}$$

$$\begin{aligned}\kappa_{7,i,2}(\rho)=&\frac{1}{2}\log\left(b_{1,i}(\rho)+1\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{ij}\right)\\&-\frac{1}{2}\log\left(1+b_{5,j}(\rho)\right)+\frac{1}{2}\log\left(1+b_{4,i}(\rho)+b_{5,j}(\rho)\right)\\&+\frac{1}{2}\log\left(1+\left(1-\rho^2\right)\frac{\mathrm{INR}_{ji}}{\overrightarrow{\mathrm{SNR}}_j}\left(\mathrm{INR}_{ij}+\right.\right.\\&\left.\left.\frac{b_{3,j}\overleftarrow{\mathrm{SNR}}_j}{b_{1,j}(1)+1}\right)\right)-\frac{1}{2}\log\left(1+\frac{b_{5,i}(\rho)\mathrm{INR}_{ji}}{\overrightarrow{\mathrm{SNR}}_j}\right)\\&+\frac{1}{2}\log\left(b_{6,j}(\rho)+\frac{b_{5,i}(\rho)\mathrm{INR}_{ji}}{\overrightarrow{\mathrm{SNR}}_j}\left(\overrightarrow{\mathrm{SNR}}_j+b_{3,j}\right)\right)\\&+2\log(2\pi e),\end{aligned}\tag{16b}$$

where the functions $b_{l,i}$, with $(l,i) \in \{1,2\}^2$ are defined in (11); $b_{3,i}$ are constants; and the functions $b_{l,i} : [0,1] \rightarrow \mathbb{R}_+$, with $(l,i) \in \{4,5,6\} \times \{1,2\}$ are defined as follows, with $j \in \{1,2\} \setminus \{i\}$:

$$b_{3,i}=\overrightarrow{\mathrm{SNR}}_i-2\sqrt{\overrightarrow{\mathrm{SNR}}_i\mathrm{INR}_{ji}}+\mathrm{INR}_{ji},\tag{17a}$$

$$b_{4,i}(\rho)=\left(1-\rho^2\right)\overrightarrow{\mathrm{SNR}}_i,\tag{17b}$$

$$b_{5,i}(\rho)=\left(1-\rho^2\right)\mathrm{INR}_{ij},\tag{17c}$$

$$\begin{aligned}b_{6,i}(\rho)=&\overrightarrow{\mathrm{SNR}}_i+\mathrm{INR}_{ij}+2\rho\sqrt{\mathrm{INR}_{ij}}\left(\sqrt{\overrightarrow{\mathrm{SNR}}_i}-\sqrt{\mathrm{INR}_{ji}}\right)\\&+\frac{\mathrm{INR}_{ij}\sqrt{\mathrm{INR}_{ji}}}{\overrightarrow{\mathrm{SNR}}_i}\left(\sqrt{\mathrm{INR}_{ji}}-2\sqrt{\overrightarrow{\mathrm{SNR}}_i}\right).\end{aligned}\tag{17d}$$

Note that the functions in (14), (15), (16) and (17) depend on $\overrightarrow{\mathrm{SNR}}_1$, $\overrightarrow{\mathrm{SNR}}_2$, INR_{12}, INR_{21}, $\overleftarrow{\mathrm{SNR}}_1$, and $\overleftarrow{\mathrm{SNR}}_2$. However, these parameters are fixed in this analysis, and therefore, this dependence is not emphasized in the definition of these functions. Finally, using this notation, Theorem 2 is presented below.

Theorem 2: The capacity region $\mathcal{C}_{\mathrm{GIC-NOF}}$ is contained within the region $\overline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$ given by the closure of the set of non-negative rate pairs (R_1, R_2) that for all $i \in \{1,2\}$, with $j \in \{1,2\} \setminus \{i\}$ satisfy:

$$R_i \leqslant \min\left(\kappa_{1,i}(\rho), \kappa_{2,i}(\rho)\right),\tag{18a}$$

$$R_i \leqslant \kappa_{3,i}(\rho),\tag{18b}$$

$$R_1+R_2 \leqslant \min\left(\kappa_4(\rho), \kappa_5(\rho)\right),\tag{18c}$$

$$R_1+R_2 \leqslant \kappa_6(\rho),\tag{18d}$$

$$2R_i+R_j \leqslant \kappa_{7,i}(\rho),\tag{18e}$$

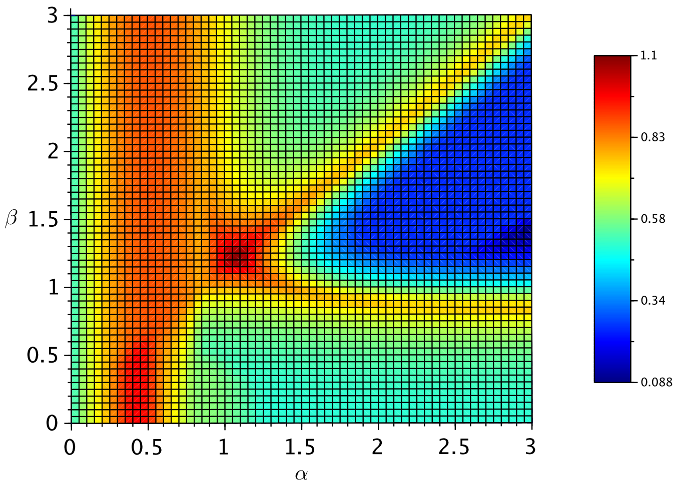

Fig. 3. Gap between the converse region $\overline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$ and the achievable region $\underline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$ of the two-user G-IC-NOF, under symmetric channel conditions, i.e., $\overrightarrow{\mathrm{SNR}}_1 = \overrightarrow{\mathrm{SNR}}_2 = \overrightarrow{\mathrm{SNR}}$, $\mathrm{INR}_{12} = \mathrm{INR}_{21} = \mathrm{INR}$, and $\overleftarrow{\mathrm{SNR}}_1 = \overleftarrow{\mathrm{SNR}}_2 = \overleftarrow{\mathrm{SNR}}$, as a function of $\alpha = \frac{\log \mathrm{INR}}{\log \overrightarrow{\mathrm{SNR}}}$ and $\beta = \frac{\log \overleftarrow{\mathrm{SNR}}}{\log \overrightarrow{\mathrm{SNR}}}$.

with $\rho \in [0, 1]$.

Proof: The proof of Theorem 2 is presented in [1]. ■

D. Comments on the Converse Region

The outer bounds (18a) and (18c) correspond to the outer bounds for the case of perfect channel-output feedback [4]. The bounds (18b), (18d) and (18e) correspond to new outer bounds that generalize those presented in [2] for the two-user symmetric G-IC-NOF. These new outer-bounds were obtained using the genie-aided models shown in Figure 2.

E. A Gap Between the Achievable Region and the Converse Region

Theorem 3 describes the gap between the achievable region $\underline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$ and the converse region $\overline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$ using the approximation notion described in Definition 2.

Theorem 3: The capacity region of the two-user G-IC-NOF is approximated to within 4.4 *bits per channel use by the achievable region* $\underline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$ *and the converse region* $\overline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$.

Proof: The proof of Theorem 3 is presented in [1]. ■

The gap, denoted by δ, between the sets $\overline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$ and $\underline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$ can be approximated (Definition 2) as follows:

$$\delta \leqslant \max\left(\delta_{R_1}, \delta_{R_2}, \frac{\delta_{2R}}{2}, \frac{\delta_{3R_1}}{3}, \frac{\delta_{3R_2}}{3}\right), \tag{19}$$

where

$$\begin{aligned}\delta_{R_1} \triangleq\ & \min\Big(\kappa_{1,1}(\rho), \kappa_{2,1}(\rho), \kappa_{3,1}(\rho)\Big) - \min\Big(a_{2,1}(\rho),\\ & a_{6,1}(\rho,\mu_1)+a_{3,2}(\rho,\mu_1), a_{1,1}+a_{3,2}(\rho,\mu_1)+a_{4,2}(\rho,\mu_1)\Big),\end{aligned} \tag{20a}$$

$$\begin{aligned}\delta_{R_2} \triangleq\ & \min\Big(\kappa_{1,2}(\rho), \kappa_{2,2}(\rho), \kappa_{3,2}(\rho)\Big) - \min\Big(a_{2,2}(\rho),\\ & a_{3,1}(\rho,\mu_2)+a_{6,2}(\rho,\mu_2), a_{3,1}(\rho,\mu_2)+a_{4,1}(\rho,\mu_2)+a_{1,2}\Big),\end{aligned} \tag{20b}$$

$$\begin{aligned}\delta_{2R} \triangleq\ & \min\Big(\kappa_4(\rho), \kappa_5(\rho), \kappa_6(\rho)\Big) - \min\Big(a_{2,1}(\rho)+a_{1,2},\\ & a_{1,1}+a_{2,2}(\rho),\\ & a_{3,1}(\rho,\mu_2)+a_{1,1}+a_{3,2}(\rho,\mu_1)+a_{7,2}(\rho,\mu_1,\mu_2),\\ & a_{3,1}(\rho,\mu_2)+a_{5,1}(\rho,\mu_2)+a_{3,2}(\rho,\mu_1)+a_{5,2}(\rho,\mu_1),\\ & a_{3,1}(\rho,\mu_2)+a_{7,1}(\rho,\mu_1,\mu_2)+a_{3,2}(\rho,\mu_1)+a_{1,2}\Big),\end{aligned} \tag{20c}$$

$$\begin{aligned}\delta_{3R_1} \triangleq\ & \kappa_{7,1}(\rho) - \min\Big(a_{2,1}(\rho)+a_{1,1}+a_{3,2}(\rho,\mu_1)\\ & +a_{7,2}(\rho,\mu_1,\mu_2), a_{3,1}(\rho,\mu_2)+a_{1,1}+a_{7,1}(\rho,\mu_1,\mu_2)\\ & +2a_{3,2}(\rho,\mu_1)+a_{5,2}(\rho,\mu_1), a_{2,1}(\rho)+a_{1,1}+a_{3,2}(\rho,\mu_1)\\ & +a_{5,2}(\rho,\mu_1)\Big),\end{aligned} \tag{20d}$$

$$\begin{aligned}\delta_{3R_2} \triangleq\ & \kappa_{7,2}(\rho) - \min\Big(a_{3,1}(\rho,\mu_2)+a_{5,1}(\rho,\mu_2)+a_{2,2}(\rho)\\ & +a_{1,2}, a_{3,1}(\rho,\mu_2)+a_{7,1}(\rho,\mu_1,\mu_2)+a_{2,2}(\rho)+a_{1,2},\\ & 2a_{3,1}(\rho,\mu_2)+a_{5,1}(\rho,\mu_2)+a_{3,2}(\rho,\mu_1)+a_{1,2}\\ & +a_{7,2}(\rho,\mu_1,\mu_2)\Big).\end{aligned} \tag{20e}$$

Note that δ_{R_1} and δ_{R_2} represent the gap between the active achievable single-rate bound and the active converse single-rate bound; δ_{2R} represents the gap between the active achievable sum-rate bound and the active converse sum-rate bound; and, δ_{3R_1} and δ_{3R_2} represent the gap between the active achievable weighted sum-rate bound and the active converse weighted sum-rate bound.

Finally, it is important to highlight that, as suggested in [2], [4], and [5], the gap between $\underline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$ and $\overline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$ can be calculated more precisely. However, the choice in (19) eases the calculations at the expense of less precision.

Figure 3 presents the exact gap existing between the achievable region $\underline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$ and the converse region $\overline{\mathcal{C}}_{\mathrm{G-IC-NOF}}$ for the case in which $\overrightarrow{\mathrm{SNR}}_1 = \overrightarrow{\mathrm{SNR}}_2 = \overrightarrow{\mathrm{SNR}}$, $\mathrm{INR}_{12} = \mathrm{INR}_{21} = \mathrm{INR}$, and $\overleftarrow{\mathrm{SNR}}_1 = \overleftarrow{\mathrm{SNR}}_2 = \overleftarrow{\mathrm{SNR}}$ as a function of $\alpha = \frac{\log \mathrm{INR}}{\log \overrightarrow{\mathrm{SNR}}}$ and $\beta = \frac{\log \overleftarrow{\mathrm{SNR}}}{\log \overrightarrow{\mathrm{SNR}}}$. Note that in this case, the maximum gap is 1.1 bits per channel use and occurs when $\alpha = 1.05$ and $\beta = 1.2$.

IV. Conclusions

An achievable region and a converse region for the two-user G-IC-NOF have been introduced. It has been shown that these regions approximate the capacity region of the two-user G-IC-NOF to within 4.4 bits per channel use.

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